Journal of Applied Mathematics and Mechanics 73 (2009) 710–721

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Redistribution of finite elastic strains after the formation of inclusions. Approximate analytical solution☆

K.M. Zingerman, V.A. Levin

Moscow, Russia

ARTICLE INFO

Article history:
Received 6 May 2008

ABSTRACT

A class of two-dimensional static problems of the stress-strain state of non-linearly elastic bodies, in which domains with different elastic properties (inclusions) arise after preloading, is considered. Problems are formulated and solved using the theory of the repeated superposition of finite strains. The mechanical properties of the initial material and the material of the inclusions are described by Murnaghan-type or Mooney-type constitutive relations. Two ways of specifying the constitutive relations for the material of an inclusion are considered: when there are inherent strains in this material and when there are not. Approximate analytical methods are used for the solution.

The problem of the stress-strain state in the neighbourhood of elastic inclusions (domains with different mechanical properties) which arise, after loading, in a body made of a non-linearly elastic compressible or incompressible material, taking account of non-linear effects due to the finiteness of the strains, can be of interest, for example, when investigating the deformation of rubber-like materials in which, when they are stretched, there is a transition from a highly elastic state into a crystalline state.[1–10] As a rule, polymers crystallize partially. At the same time, crystalline domains originate in a material that, in a number of cases, can be considered as inclusions.[11,12]. An inclusion can arise in a loaded body as the result of a phase transition.[3–16] such as, for example, in shape memory alloys.[17,18] The formation of micropores in a certain part of a body (for example, in a pre-existing imperfection zone[19]) leads to a change in the effective mechanical properties of the material in the corresponding region which can also be considered as the formation of an inclusion.[20–25]

The first problems concerning inclusions in a linearly elastic medium were considered in the middle of the last century.[26,27] Several non-linear effects have been investigated.[28,29] The problem of the phase transformation of an elastic sphere has been considered.[30] A solution has been obtained[28] in the case of small deformations, where only the material of the inclusion was assumed to be non-linearly elastic, and the material of the remaining part of the body (the matrix) was assumed to be linearly elastic. A solution has been obtained[29] for large deformations using a special form of elastic potential for which it was found to be possible to reduce the non-linear boundary value problem to a linear boundary value problem and to obtain an exact analytical solution. In all the above mentioned papers, with the exception of Ref. 30, it was assumed that the inclusion existed in the body from the very outset. It is assumed below that an inclusion is formed in a predeformed body. It is also assumed that the strains throughout the body are finite and that the matrix and inclusion materials are non-linearly elastic. The approach employed to solve the problem, based on the use of approximate analytical methods, is applicable for arbitrary elastic potentials in the case of both compressible and incompressible isotropic materials. In the case of small deformations, the problems has been solved[30] in a radially symmetric formulation for a body of finite dimensions (a sphere) and the boundaries of the domain in which the properties of the material changed were determined when solving the problem using an additional condition at the phase interface. Plane problems for infinitely extended bodies are considered below and the domain in which the properties of the material change is assumed to be given.

1. The model of the formation of the inclusion

We shall assume that the formation of an inclusion in a body occurs "instantaneously". The approach adopted in the theory of the superposition of large strains[22,31,32] can then be used in the case of static problems. The generalized formulation of the problem of the

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 6, pp. 983–1001, 2009.

E-mail addresses: zingerman@rambler.ru (K.M. Zingerman), v.a.levin@mail.ru (V.A. Levin).

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.01.011

formation of an inclusion based on this approach is as follows. Suppose large static strains and stresses, which we call the initial strains and stresses, arise in a non-linearly elastic body in an initial (unstressed) state under the action of external loads. The body has transferred into the first intermediate state. A closed surface (the future boundary of the inclusion) is then conceptually designated in this body. The part of the stressed body bounded by the designated surface is conceptually removed from the remaining part of the body and its action on the remaining part of the body is replaced, according to the principle of releasing from constraints, by forces distributed over this surface. This action does not change the stress-strain state of the remaining part of the body. The cavity which has been formed by removing a part of the body is then filled with an elastic material with different properties (the material of the inclusion). It is assumed here that forces are applied to the boundary of the inclusion. Two versions of their application are considered. The first version is: forces are applied which give rise to a deformation of the inclusion equal to the initial deformation of the removed part of the body (and the inclusion takes the shape of the removed part of the body after this deformation). The second version is: the given forces are assumed to be equal to zero and the inclusion which is inserted is at first undeformed.

The remaining part of the initial body (the matrix) and the inclusion are then "glued" while preserving the forces acting on them (the term "gluing" is understood in the sense of satisfying boundary conditions, that is, it means that, in the case of further deformation of the body, the displacements of the boundary points of the matrix will be equal to the displacements of the corresponding boundary points of the inclusion). Then, at each point of the interface between the matrix and the inclusion, the sum of the forces applied to the matrix and the forces applied to the inclusion quasi-statically (for example, isothermally) reduce to zero. This causes the onset (at least, in the matrix in the neighbourhood of the inclusion and in the inclusion itself) of large strains and stresses superimposed on the existing strains and stresses which are already large. The shape of the boundary between the matrix and the inclusion changes. The body (the matrix and the inclusion) transfers into the final state.

Note that, in the case of the first version of the model of the formation of the inclusion, the stresses in the inclusion after its deformation are determined by the overall deformations in it (these overall deformations are the superposition of the initial deformations of the part of the body removed and the additional deformations caused by the formation of the inclusion). In this case, after removal of the loads, the body (the matrix and the inclusion) transfers into the initial undeformed state and residual stresses do not occur. This version of the model can clearly be used to describe phase transitions of the second kind and, also, to describe the formation in a body of a region with randomly distributed micropores, the sizes of which are much smaller than the dimensions of this region.

In the case of the second model of the formation of an inclusion, the stresses in the inclusion after its deformation are determined by the additional deformations in it. In this case, after the loads have been removed, the body does not revert to the initial undeformed state, and residual strains and stresses arise in it. This version of the model can clearly be used to describe phase transitions of the first kind. In this case, not only the elastic moduli change during a phase transition in the material but inhernt strains deformations arise which, in this version of the model, are equal to the initial deformations.

The second version of the model is preferable to the first version for describing the mechanical behaviour of polymers during their partial crystallization. As was mentioned above, this is associated with the fact that, during the crystallization of polymers, their elastic moduli increase by a large factor while there is only a slight change in the deformation. If the first version of the model is used, the stresses must simultaneously increase by a large factor. However, there is no mention in the literature[2,3,6] that, the stresses in deformed polymers increase significantly during their crystallization.

The model considered can easily be extended to the case of the simultaneous or sequential formation of several inclusions. Note that, in the case of the sequential formation of closely spaced inclusions, there is a repeated superposition of large deformations.

2. Mathematical formulation of the problem

The mathematical formulation of the problem is based on the theory of the repeated superposition of large deformations in the coordinates of the intermediate or final state. It is assumed that the mechanical properties of the matrix and inclusion materials are described by constitutive relations of the Murnaghan- or Mooney-type[33] and that the formation of an inclusion does not change the stresses and strains at infinity. The mass forces are assumed to be equal to zero, the initial stresses and strains are assumed to be homogeneous, and the matrix is assumed to be unbounded.

Suppose the subscript 0 corresponds to the initial state, superscript 1 to the intermediate state and subscript 2 corresponds to the final state. The following notation is used:[22,31]

$\mathbf{u}_n$ is a displacement vector, characterizing the transition from the preceding $(n - 1)$-th state into the following n-th state ($\mathbf{u}_1$ is the initial displacement vector and $\mathbf{u}_2$ is the additional displacement vector);
$\overset{p}{\nabla}$ is a gradient;
$\Psi_{q,p}$ is the deformation gradient for the transition of the body from state q into state p ($\Psi_{0,1}$ is the initial deformation gradient and $\Psi_{1,2}$ is the additional deformation gradient).;
$\Delta_{m,n}$ is the relative change in volume on transferring from the m-th into the n-th state:
$\sigma_{0,n}$ is the true stress tensor, describing the stresses accumulated in the body on transferring from the initial into the n-th state;
$p_{0,n}$ is the Lagrange multiplier for the n-th state (for incompressible materials);
$\overset{k}{\mathrm{E}}_{m,n}$ is the strain tensor in transferring from the m-th into the n-th state, referred to the basis of the k-th state;
$\mathrm{F}_{m,n}$ is the strain measure tensor on transferring from the m-th to the n-th state corresponding to the Finger measure;
$\overset{m}{\Sigma}_{0,n}$ is the generalized (overall for the n-th state) stress tensor defined in the coordinate basis of an arbitrary m-th state;
$\Gamma^{(j)}$ is the boundary of the i-th contour ($i = 1,\ldots, L$, L is the number of inclusions) in the coordinates of the intermediate state;
I is a second rank unit tensor;
N is the normal to the boundary of the body or to the boundary of an inclusion in the coordinates of the k-th state;
$S^{(i)}$ ($i = 1,\ldots,L$) is the region, the outer boundary of which is the contour $\Gamma^{(i)}$;
$S^{(0)}$ is the infinite domain bounded by the contours $\Gamma^{(i)}$ ($i = 1,\ldots,L$).

It is well known[22,31] that, before solving the problem for the final state in the coordinates of the intermediate state, it is necessary to find the initial displacement vector $\mathbf{u}_1$ or the initial deformation gradient $\Psi_{0,1}$, that is, to solve the problem for the intermediate state in the coordinates of this state. Note that, as a consequence of the homogeneity of the material of the body in the initial state, the initial strains will be homogeneous. The problem of finding the initial strains is simple and is not considered in this paper.

We well now consider the formulation of the problem for the final state in the coordinates of the intermediate or final state. The equilibrium equation for solving the problem in the coordinates of the intermediate state can be written in the form

$$\overset{1}{\nabla}\cdot[(1+\Delta_{0,1})^{-1}\overset{1}{\mathbf{\Sigma}}_{0,2}\cdot\mathbf{\Psi}_{1,2}]=0 \tag{2.1}$$

Here,

$$\overset{1}{\mathbf{\Sigma}}_{0,2}=(1+\Delta_{0,2})\mathbf{\Psi}^{*-1}_{1,2}\cdot\sigma_{0,2}\cdot\mathbf{\Psi}^{-1}_{1,2} \tag{2.2}$$

When solving the problem in the coordinates of the final state, the equilibrium equation has the form

$$\overset{2}{\nabla}\cdot\sigma_{0,2}=0 \tag{2.3}$$

For an in compressible material (a Mooney material) the condition

$$\Delta_{0,n}=0 \tag{2.4}$$

must be satisfied.

The constitutive relations for the initial material (the matrix material) are written in a form corresponding to the Murnaghan potential

$$\begin{aligned}\overset{0}{\mathbf{\Sigma}}_{0,2}&=\lambda_M(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})\mathbf{I}+2G_M\overset{0}{\mathbf{E}}_{0,2}+3C_3^M(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})^2\mathbf{I}+\\&+C_4^M(\overset{0}{\mathbf{E}}{}^{2}_{0,2}:\mathbf{I})\mathbf{I}+2C_4^M(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})\overset{0}{\mathbf{E}}_{0,2}+3C_5^M(\overset{0}{\mathbf{E}}_{0,2})^2\end{aligned} \tag{2.5}$$

or the Mooney potential

$$\sigma_{0,2}=\frac{G_M}{2}\{(1+\beta_M)\mathbf{F}_{0,2}+(1-\beta_M)[(\mathbf{F}_{0,2}:\mathbf{I})\mathbf{F}_{0,2}-\mathbf{F}^2_{0,2}]\}-p_{0,2}\mathbf{I} \tag{2.6}$$

The constitutive relations for an inclusion are written in a similar form.

In the case of the first version of the model of an inclusion, the constitutive relations have the form

$$\begin{aligned}\overset{0}{\mathbf{\Sigma}}_{0,2}&=\lambda_B(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})\mathbf{I}+2G_B\overset{0}{\mathbf{E}}_{0,2}+3C_3^B(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})^2\mathbf{I}+\\&+C_4^B(\overset{0}{\mathbf{E}}{}^{2}_{0,2}:\mathbf{I})\mathbf{I}+2C_4^B(\overset{0}{\mathbf{E}}_{0,2}:\mathbf{I})\overset{0}{\mathbf{E}}_{0,2}+3C_5^B(\overset{0}{\mathbf{E}}_{0,2})^2\end{aligned} \tag{2.7}$$

for the Murnaghan potential or

$$\sigma_{0,2}=\frac{G_B}{2}\{(1+\beta_B)\mathbf{F}_{0,2}+(1-\beta_B)[(\mathbf{F}_{0,2}:\mathbf{I})\mathbf{F}_{0,2}-\mathbf{F}^2_{0,2}]\}-p_{0,2}\mathbf{I} \tag{2.8}$$

for the Mooney potential. Henceforth, the index M refers to the matrix and the index B refers to the inclusion. The constants $\lambda_M, G_M, C_3^M, C_4^M, C_5^M, \beta_M$ characterize the elastic properties of the matrix material (λ_M and G_M are Lamé constants) and $\lambda_B, G_B, C_3^B, C_4^B, C_5^B, \beta_B$ determine the elastic properties of the inclusion material.

In the case of the second version of the model for the formation of an inclusion (when, at the instant of "insertion", the inclusion is not deformed), the constitutive relations have the form

$$\begin{aligned}\overset{0}{\mathbf{\Sigma}}_{0,2}&=\lambda_B(\overset{0}{\mathbf{E}}_{1,2}:\mathbf{I})\mathbf{I}+2G_B\overset{0}{\mathbf{E}}_{1,2}+3C_3^B(\overset{0}{\mathbf{E}}_{1,2}:\mathbf{I})^2\mathbf{I}+\\&+C_4^B(\overset{0}{\mathbf{E}}{}^{2}_{1,2}:\mathbf{I})\mathbf{I}+2C_4^B(\overset{0}{\mathbf{E}}_{1,2}:\mathbf{I})\overset{0}{\mathbf{E}}_{1,2}+3C_5^B(\overset{0}{\mathbf{E}}_{1,2})^2\end{aligned} \tag{2.9}$$

in the case of the Murnaghan potential or

$$\sigma_{0,2}=\frac{G_B}{2}\{(1+\beta_B)\mathbf{F}_{1,2}+(1-\beta_B)[(\mathbf{F}_{1,2}:\mathbf{I})\mathbf{F}_{1,2}-\mathbf{F}^2_{1,2}]\}-p_{0,2}\mathbf{I} \tag{2.10}$$

in the case of the Mooney potential.

Relations (2.5) or (2.6) hold in the domain $S^{(0)}$ and relations (2.7)–(2.10) hold in the domains $S^{(i)}$ ($i=1,\ldots,L$). In these relations,

$$\overset{0}{\mathbf{\Sigma}}_{0,2}=\mathbf{\Psi}^{*-1}_{0,1}\cdot\overset{1}{\mathbf{\Sigma}}_{0,2}\cdot\mathbf{\Psi}^{-1}_{0,1} \tag{2.11}$$

and, consequently,

$$\overset{1}{\mathbf{\Sigma}}_{0,2}=\mathbf{\Psi}^{*}_{0,1}\cdot\overset{0}{\mathbf{\Sigma}}_{0,2}\cdot\mathbf{\Psi}_{0,1} \tag{2.12}$$

The kinematic relations can be written in the following form:[22]

$$\overset{0}{\mathbf{E}}_{0,2} = \frac{1}{2}(\Psi_{0,2}\cdot\Psi^*_{0,2}-\mathbf{I}),\quad \overset{1}{\mathbf{E}}_{1,2} = \frac{1}{2}(\Psi_{1,2}\cdot\Psi^*_{1,2}-\mathbf{I}) \tag{2.13}$$

$$(1+\Delta_{0,2}) = \det\mathbf{\Psi}_{0,2},\quad (1+\Delta_{1,2}) = \det\mathbf{\Psi}_{1,2},\quad \mathbf{\Psi}_{0,2} = \mathbf{\Psi}_{0,1}\cdot\mathbf{\Psi}_{1,2} \tag{2.15}$$

$$\mathbf{F}_{0,2} = \mathbf{\Psi}^*_{0,2}\cdot\mathbf{\Psi}_{0,2},\quad \mathbf{F}_{1,2} = \mathbf{\Psi}^*_{1,2}\cdot\mathbf{\Psi}_{1,2} \tag{2.14}$$

$$(1+\Delta_{0,2}) = \det\mathbf{\Psi}_{0,2},\quad (1+\Delta_{1,2}) = \det\mathbf{\Psi}_{1,2},\quad \mathbf{\Psi}_{0,2} = \mathbf{\Psi}_{0,1}\cdot\mathbf{\Psi}_{1,2} \tag{2.16}$$

$$\mathbf{\Psi}_{1,2} = \mathbf{I} + \overset{1}{\nabla}\mathbf{u}_2 \tag{2.16}$$

$$\mathbf{\Psi}_{1,2} = \left(\mathbf{I} - \overset{2}{\nabla}\mathbf{u}_2\right)^{-1} \tag{2.17}$$

Formula (2.16) is used when solving the problem in the coordinates of the intermediate state, and formula (2.17) when solving the problem in the coordinates of the final state.

The boundary conditions include the condition at infinity

$$\mathbf{\sigma}_{0,2}\big|_{\infty} = \mathbf{\sigma}^{\infty} \tag{2.18}$$

(σ^{∞} is the given tensor of the true stresses at infinity) and, also, conditions for the continuity of the displacement vector $\mathbf{u}_2$ and the normal stress vector $N_1\cdot\overset{1}{\Sigma}_{0,2}\cdot\Psi_{1,2}$ at the boundaries of the inclusions, that is, on the contours $\Gamma^{(i)}$. Denoting the increment in a certain quantity on passing across an interface $\Gamma^{(i)}$ by $[[\cdot]]_{\Gamma^{(i)}}$, these conditions can be written in the following form:

$$[[\mathbf{u}_2]]_{\Gamma^{(i)}} = \mathbf{0},\quad i = 1,\ldots,L \tag{2.19}$$

$$\mathbf{N}_1\cdot\left[\left[\overset{1}{\mathbf{\Sigma}}_{0,2}\cdot\mathbf{\Psi}_{1,2}\right]\right]_{\Gamma^{(i)}} = \mathbf{0},\quad i = 1,\ldots,L \tag{2.20}$$

Condition (2.20) can be derived from the corresponding condition in the coordinates of the final state

$$\mathbf{N}_2\cdot[[\mathbf{\sigma}_{0,2}]]_{\gamma^{(i)}} = \mathbf{0},\quad i = 1,\ldots,L \tag{2.21}$$

The contour into which the boundary $\Gamma^{(i)}$ transfers when the body charges from the intermediate state into the final state is denoted by $\gamma^{(i)}$.

To derive formula (2.20), we transform relation (2.21). We first multiply it by the area of an elementary small area in the final state dO_2:

$$[[\mathbf{N}_2 dO_2\cdot\mathbf{\sigma}_{0,2}]]_{\gamma} = \mathbf{0} \tag{2.22}$$

(the superscript i is omitted for brevity).

We now express the tensor $\sigma_{0,2}$ in terms of $\overset{1}{\Sigma}_{0,2}$ using equality (2.2) and substitute it into relation (2.22). We also use the identity[25]

$$\mathbf{N}_2 dO_2 = (1+\Delta_{1,2})\mathbf{N}_1 dO_1\cdot\mathbf{\Psi}^{*-1}_{1,2} \tag{2.23}$$

which is the generalization of the well known formula[33]

$$\mathbf{N}dO = (1+\Delta)\mathbf{\Psi}^{-1}\cdot\mathbf{n}do = (1+\Delta)\mathbf{n}\cdot\mathbf{\Psi}^{*-1}do$$

to the case of the superposition of large deformations, where **N** and **n** are the vectors of the unit normals to the orientated small area in the deformed and undeformed states respectively, dO and do are the areas of this small area in the corresponding states, Ψ is the deformation gradient and Δ is the relative change in volume.

As a result of the substitutions, formula (2.22) can be rewritten in the intermediate state coordinates in the form

$$[[(1+\Delta_{1,2})(1+\Delta_{0,2})^{-1}\mathbf{N}_1 dO_1\cdot\mathbf{\Psi}^{*-1}_{1,2}\cdot\mathbf{\Psi}^*_{1,2}\cdot\overset{1}{\mathbf{\Sigma}}_{0,2}\cdot\mathbf{\Psi}_{1,2}]]_{\Gamma} = \mathbf{0} \tag{2.24}$$

whence

$$\mathbf{N}_1 dO_1\cdot[[(1+\Delta_{0,1})^{-1}\overset{1}{\mathbf{\Sigma}}_{0,n}\cdot\mathbf{\Psi}_{1,2}]]_{\Gamma} = \mathbf{0} \tag{2.25}$$

We will now take account of the fact that, before the change in the properties of the material, the strains in the body were continuous and it is therefore possible to take the scalar $(1+\Delta_{0,1})^{-1}$ out of the brackets. Dividing equality (2.25) by this quantity and dO_1, we finally obtain the boundary condition in the form of (2.20). Note that this form of writing the boundary condition is a generalization to the case of the superposition of the residual strains of the boundary condition in the initial state coordinates (for example, see Refs. 14 and 34).

3. Solution of the problem using approximate analytical methods

In the case of plane deformation and a circular inclusion (at the instant of formation or in the final state), the problem can be solved using approximate analytical methods. The small-parameter method (the method of successive approximations)[31,33,35] and the modified Newton–Kantorovich method[31,36] are used to solve it. The essence of the small parameter method, as applied to the problems considered here, is as follows. A small parameter is written in the form

$$q = \max_{i,j} |(\sigma^{\infty})_{ij}| / G_M \tag{3.1}$$

and a series expansion in this parameter is written for all the quantities occurring in the formulation of the problem. For example, in the case of the displacement vector $\mathbf{u}_2$, this expansion can be written in the form

$$\mathbf{u}_2 = \mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)} + \ldots, \quad \mathbf{u}_2^{(i)} \sim q^{i+1} \tag{3.2}$$

After substituting similar expansions into all the equations appearing in the formulation of the problem, the solution of the initial non-linear problem reduces to the successive solution of linearized problems.

The small parameter method is applicable in the case when the strains are finite but small compared with unity (although, for certain boundary value problems and classes of materials, this method converges over a wider range of strains[37]). It has been noted[33] that this method, when terms of the second order with respect to the small parameter are retained can be used to take account of the quantitative deviations from the predictions of the linear theory.

In the case of several inclusions, the linearized problem for each approximation is solved using an iterative algorithm.[31,38] At each step in the iterative algorithm, the problem for one inclusion is solved with boundary conditions specified in a special way. The method of solving this problem using complex Kolosov–Muskhelishvili potentials[27,39] is described in detail in the following Section. A specialized system of numerical-analytical computations using a computer[31] is employed in the calculations. The first two approximations (linear and quadratic) were calculated by the small parameter method. Five approximations for problems involving one inclusion were found by the Newton–Kantorovich method.

4. Analytical solution of the linearized problem for one inclusion

The linearized problem of a circular elastic inclusion in an infinitely extended elastic body (when there are no mass forces)[40] is considered for the case of plane deformation. The formulation of the problem includes the equilibrium equations for the matrix and the inclusion

$$\nabla \cdot \boldsymbol{\sigma}^M = 0, \quad \nabla \cdot \boldsymbol{\sigma}^B = 0 \tag{4.1}$$

the boundary conditions

$$\mathbf{N} \cdot \boldsymbol{\sigma}^M|_\Gamma = \mathbf{N} \cdot \boldsymbol{\sigma}^B|_\Gamma + \mathbf{Q}, \quad \mathbf{u}^M|_\Gamma = \mathbf{u}^B|_\Gamma + \tilde{\mathbf{u}} \tag{4.2}$$

the condition at infinity

$$\boldsymbol{\sigma}^M|_\infty = 0 \tag{4.3}$$

the constitutive relations for the matrix and the inclusion

$$\boldsymbol{\sigma}^M = \lambda_M(\varepsilon^M : \mathbf{I})\mathbf{I} + 2G_M\varepsilon^M, \quad \boldsymbol{\sigma}^B = \lambda_B(\varepsilon^B : \mathbf{I})\mathbf{I} + 2G_B\varepsilon^B \tag{4.4}$$

and the kinematic relations for the matrix and the inclusion

$$\varepsilon^M = \frac{1}{2}(\nabla\mathbf{u}^{\mathbf{M}} + \mathbf{u}^M\nabla), \quad \varepsilon^B = \frac{1}{2}(\nabla\mathbf{u}^B + \mathbf{u}^B\nabla) \tag{4.5}$$

Here, $\mathbf{u}^M$ and $\mathbf{u}^B$ are the displacement vectors in the matrix and in the inclusion respectively (or the increments in the displacement vectors), ε^M and ε^B are the tensors for small strains in the matrix and in the inclusion (or increments in the strain tensors), σ^M and σ^B are the linear elasticity stress tensors in the matrix and in the inclusion (or increments in the stress tensor), Γ is the boundary of the inclusion, $\mathbf{N}$ is the vector of the normal to this boundary, and $\mathbf{Q}$ and $\tilde{u}$ are the residual vectors with respect to the normal stresses and with respect to the displacements on the boundary, respectively.

This problem arises, for example, when the small parameter method is used to solve a non-linear elasticity problem at the stage of finding the solution of the homogeneous equation satisfying the boundary conditions. The problem that arises at each step in the Schwartz method when calculating of several interacting inclusions in an infinitely extended body, has the same form.

We note that a more general problem, in which the condition at infinity (4.3) is replaced by the condition $\sigma^M|_\infty = \sigma^\infty$ can be reduced to the problem formulated above by representing the stress field in the form of the sum of two fields: a field of constant stresses equal to σ^∞ and a field of additional stresses satisfying conditions (4.3).

Complex Kolosov–Muskhelishvili potentials[27] can be used to solve the plane problem. The complex potentials for the matrix are denoted by $\varphi_M(z)$ and $\psi_M(z)$ and the complex potentials for the inclusion by $\varphi_B(z)$ and $\psi_M(z)$. The components of the stress tensor σ^ξ and the displacement vector u^ξ ($\xi = M$ for the matrix and $\xi = B$ for the inclusion) are related to the complex potentials by the equations

$$u_1^\xi + iu_2^\xi = \frac{1}{2G_\xi}\left[æ_\xi\varphi_\xi(z) - z\overline{\varphi'_\xi(z)} - \overline{\psi_\xi(z)}\right], \quad \xi = M, B \tag{4.6}$$

$$\sigma_{11}^{\xi} + \sigma_{22}^{\xi} = 4\Re\varphi_{\xi}'(z), \quad \sigma_{22}^{\xi} - \sigma_{11}^{\xi} + 2i\sigma_{12}^{\xi} = 2\left[\bar{z}\varphi_{\xi}''(z) + \psi_{\xi}'(z)\right] \tag{4.7}$$

where

$$æ_{\xi} = \frac{\lambda_{\xi} + 3G_{\xi}}{\lambda_{\xi} + G_{\xi}}$$

Henceforth, z denotes an arbitrary point in the complex plane. We shall denote an arbitrary point of the contour Γ (the boundary of the inclusion) by $t = e^{i\theta}$, the finite region bounded by the contour Γ (the domain of the inclusion) by S^+ and the infinite region bounded by this contour (the domain of the matrix) by S^-.

We shall further assume that the boundary of the inclusion is a unit circle with centre at the origin of coordinates. The solution for a circular inclusion of arbitrary radius with an arbitrary centre can be obtained by the same method as above but the calculations are more unwieldy.

Taking account of relations (4.7), the first of boundary conditions (4.2) can be written, after integration along the contour, in the form[27]

$$\left[\varphi_M(t) + t\overline{\varphi_M'(t)} + \overline{\psi_M(t)}\right]_{\Gamma} = \left[\varphi_B(t) + t\overline{\varphi_B'(t)} + \overline{\psi_B(t)}\right]_{\Gamma} + g(t) \tag{4.8}$$

where

$$g(t) = \int_{\theta_0}^{\theta} q\,d\alpha = \int_{t_0}^{t} q(s)\frac{ds}{s}$$

Here $t_0 = e^{i\theta_0}$ and $s = e^{i\alpha}$ are certain points of the contour Γ and $q = Q_1 + iQ_2$ is the complex representation of the residual vector with respect to the normal stresses **Q**.

When account is taken of relations (4.6), the second boundary condition of (4.2) can be written in the form

$$\frac{1}{2G_M}\left[æ_M\varphi_M(t) - t\overline{\varphi_M'(t)} - \overline{\psi_M(t)}\right]_{\Gamma} = \frac{1}{2G_B}\left[æ_B\varphi_B(t)\right] t\overline{\varphi_B'(t)} - \overline{\psi_B(t)}\Big]_{\Gamma} + h(t) \tag{4.9}$$

Here $h(t) = \tilde{u}_1 + i\tilde{u}_2$ is the complex representation of the residual vector with respect to the displacements $\tilde{\mathrm{u}}$.

We shall assume that the functions $g(t)$ and $h(t)$ can be expanded in series

$$g(t) = \sum_{k=-\infty}^{\infty} g_k t^k, \quad h(t) = \sum_{k=-\infty}^{\infty} h_k t^k \tag{4.10}$$

The complex potentials $\varphi_M(z)$ and $\psi_M(z)$ are analytic functions in the infinite domain $|z| > 1$,[27] and the expansions in series

$$\varphi_M(z) = \sum_{k=0}^{\infty} a_k z^{-k}, \quad \psi_M(z) = \sum_{k=0}^{\infty} b_k z^{-k} \tag{4.11}$$

hold for them.

The complex potentials $\varphi_B(z)$ and $\psi_B(z)$ are analytic functions within the unit circle (when $|z| < 1$) and can be represented in the form of expansions in non-negative powers of z

$$\varphi_B(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \psi_B(z) = \sum_{k=0}^{\infty} d_k z^k \tag{4.12}$$

To find the solution, we will use Cauchy-type integrals. We represent the solution in the form

$$\begin{gathered}\varphi_M(z) = a_0 + \overset{*}{\varphi}_M(z), \quad \psi_M(z) = b_0 + b_1 z^{-1} + \overset{*}{\psi}_M(z)\\ \varphi_B(z) = c_0 + c_1 z + \overset{*}{\varphi}_B(z), \quad \psi_B(z) = d_0 + \overset{*}{\psi}_B(z)\end{gathered} \tag{4.13}$$

The functions $g(t)$ and $h(t)$ are represented in the form

$$g(t) = g_0 + g_1 t + g^*(t), \quad h(t) = h_0 + h_1 t + h^*(t) \tag{4.14}$$

The functions $\varphi_M^*(z)$, $\psi_M^*(z)$, $\varphi_B^*(z)$, $\psi_B^*(z)$ can be determined from the boundary conditions

$$\begin{gathered}\left[\overset{*}{\varphi}_M(t) + t\overline{\overset{*}{\varphi}_{M'}(t)} + \overline{\overset{*}{\psi}_M(t)}\right]_{\Gamma} = \left[\overset{*}{\varphi}_B(t) + t\overline{\overset{*}{\varphi}_{B'}(t)} + \overline{\overset{*}{\psi}_B(t)}\right]_{\Gamma} + g^*(t)\\ \frac{1}{2G_M}\left[æ_M\overset{*}{\varphi}_M(t) - t\overline{\overset{*}{\varphi}_M(t)} - \overline{\overset{*}{\psi}_{M'}(t)}\right]_{\Gamma} = \frac{1}{2G_B}\left[æ_B\overset{*}{\varphi}_B(t) - t\overline{\overset{*}{\varphi}_{B'}(t)} - \overline{\overset{*}{\psi}_B(t)}\right]_{\Gamma} + h^*(t)\end{gathered} \tag{4.15}$$

We introduce the notation

$$I_1(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{g^*(t)dt}{t-z}, \quad I_2(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{h^*(t)dt}{t-z}$$

$$\tilde{I}_1(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\overline{g^*(t)}dt}{t-z}, \quad \tilde{I}_2(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\overline{h^*(t)}dt}{t-z}$$

Initially, suppose $z \in S^-$. Using the properties of Cauchy-type integrals,[27] the representations of the functions $\varphi_M(z)$, $\psi_M(z)$, $\varphi_B(z)$, $\psi_B(z)$ in the form of series and also taking account of the fact that $\bar{t} = t^{-1}$ on the contour, Eqs. (4.15) can be rewritten in the form

$$-\varphi_M^*(z) = -\left[z\overline{\varphi_B^{*\prime}}(z) + \overline{\psi_B^*}(z) - 2\bar{c}_2\right] + I_1(z) - \frac{æ_M}{2G_M}\varphi_M^*(z) = \frac{1}{2G_B}\left[z\overline{\varphi_B^{*\prime}}(z) + \overline{\psi_B^*}(z) - 2\bar{c}_2\right] + I_2(z) \tag{4.16}$$

Here, $\overline{\varphi_B^{*\prime}}(z)$ and $\overline{\psi_B^*}(z)$ are functions which are obtained by replacing $\bar{z}$ by $1/z$ in the expressions $\overline{\varphi_B^{*\prime}(z)}$ and $\overline{\psi_B^*(z)}$ respectively.

Eliminating the expression in square brackets from the system of equations (4.16), we can obtain

$$\varphi_M^*(z) = -\left(1 + \frac{G_B}{G_M}æ_M\right)^{-1}[I_1(z) + 2G_B I_2(z)] \tag{4.17}$$

Now, suppose $z \in S^+$. In this case, using the properties of Cauchy-type integrals, Eqs. (4.15) can be reduced to the form

$$z\overline{\varphi_M^{*\prime}}(z) + \overline{\psi_M^*}(z) = \varphi_B^*(z) + I_1(z) - \frac{1}{2G_M}\left[z\overline{\varphi_M^{*\prime}}(z) + \overline{\psi_M^*}(z)\right] = \frac{æ_B}{2G_B}\varphi_B^*(z) + I_2(z) \tag{4.18}$$

Here, $\overline{\varphi_M^{*\prime}}(z)$ and $\overline{\psi_M^*}(z)$ are functions which are obtained by replacing $\bar{z}$ by $1/z$ in the expressions $z\overline{\varphi_M^{*\prime}(z)}$ and $\overline{\psi_M^{*\prime}(z)}$ respectively.

Eliminating the expression $z\overline{\varphi_M^{*\prime}}(z) + \overline{\psi_M^{*\prime}}(z)$ from system of equations (4.18), we find

$$\varphi_B^*(z) = -\left(1 + \frac{G_M}{G_B}æ_B\right)^{-1}[I_1(z) + 2G_M I_2(z)] \tag{4.19}$$

In order to find the functions $\psi_M^*(z)$ and $\psi_B^*(z)$, we apply the operation of complex conjugation to the boundary conditions (4.15) and evaluate the Cauchy-type integrals of the resulting equations. In the case when $z \in S^-$, we arrive at the equations

$$\begin{aligned} &-\frac{1}{z}\varphi_M^{*\prime}(z) - \psi_M^*(z) = -\overline{\varphi_B^*}(z) + \tilde{I}_1(z) \\ &\frac{1}{2G_M}\left[\frac{1}{z}\varphi_{M'}^*(z) + \psi_M^*(z)\right] = -\frac{æ_B}{2G_B}\overline{\varphi_B^*}(z) + \tilde{I}_2(z) \end{aligned} \tag{4.20}$$

Eliminating the function $\overline{\varphi_B^{*\prime}}(z)$ from the system of equations (4.20), we can obtain

$$\psi_M^*(z) = -\frac{1}{z}\varphi_{M'}^*(z) - \left(1 + \frac{G_B}{G_M æ_B}\right)^{-1}\left[\tilde{I}_1(z) - \frac{2G_B}{æ_B}\tilde{I}_2(z)\right] \tag{4.21}$$

Finally, when $z \in S^+$, using a similar approach we find

$$\psi_B^*(z) = -\frac{1}{z}\varphi_{B'}^*(z) + \left(1 + \frac{G_M}{G_B æ_M}\right)^{-1}\left[\tilde{I}_1(z) - \frac{2G_M}{æ_M}\tilde{I}_2(z)\right] \tag{4.22}$$

Formulae (4.21) and (4.22) enable us to find the functions $\psi_M^*(z)$ and $\psi_M^*(z)$ when the functions $\varphi_M^*(z)$ and $\varphi_B^*(z)$ are already known.

In order to determine the coefficients a_0, b_0, c_0, d_0, b_1 and c_1 required to find the functions $\varphi_M(z)$, $\psi_M(z)$, $\varphi_B(z)$, $\psi_B(z)$ we substitute expansions (4.10)–(4.12) into boundary conditions (4.8) and (4.9), taking account of the equality $\bar{t} = t^{-1}$ and consider the terms containing t^0 and t^1 in the resulting series. As a result, we obtain the following equations

$$\bar{b}_1 = c_1 + \bar{c}_1 + g_1, \quad \frac{1}{2G_M}\bar{b}_1 = \frac{1}{2G_B}(æ_B c_1 + \bar{c}_1) + h_1 \tag{4.23}$$

$$a_0 + \bar{b}_0 = c_0 + 2\bar{c}_2 + \bar{d}_0 + g_0, \quad \frac{1}{2G_M}(æ_M a_0 - \bar{b}_1) = \frac{1}{2G_B}(æ_B c_0 - 2\bar{c}_2 - \bar{d}_0) + h_0 \tag{4.24}$$

Simultaneous solution of Eqs. (4.23) enables us to determine the coefficients b_1 and c_1 uniquely. The coefficients a_0, b_0, c_0 and d_0 are determined from the solution of system (4.24). Any two of them can be specified arbitrarily. For example, one can put a_0, $b_0 = 0$, and the coefficients c_0 and d_0 are then uniquely defined. A change in the values of the coefficients a_0 and b_0 corresponds to rigid translation of the whole body. Note that it is first necessary to find the coefficient c_2 in order to solve system (4.24). It can be found after the function $\varphi_B^*(z)$ has been determined.

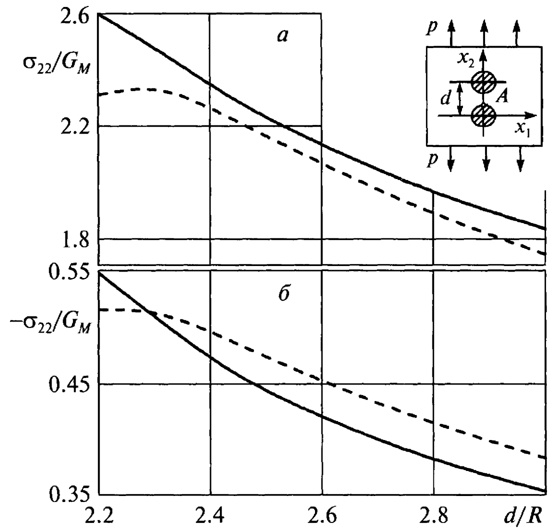

Fig. 1.

The approach described can also be used when the matrix and inclusion materials are incompressible. In this case, it is necessary to put $æ_M = 1$ and $æ_B = 1$.

5. Some results of the calculations

We will now present the results of some model problems on the formation of inclusions in a preloaded non-linearly elastic body. All calculations were carried out using the software included in the structure of the authors' suite of programs "Superposition (Nalozhenie)".[31,25] In Figs. 1–3 and 7 presented below, the solid curve corresponds to the solution obtained within the limits of the linear theory (the zeroth approximation) and the dashed curve corresponds to the solution obtained taking account of non-linear effects.

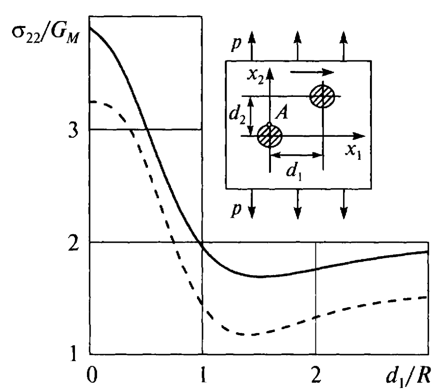

Fig. 2.

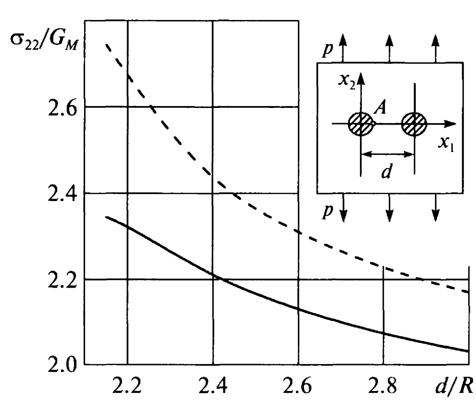

Fig. 3.

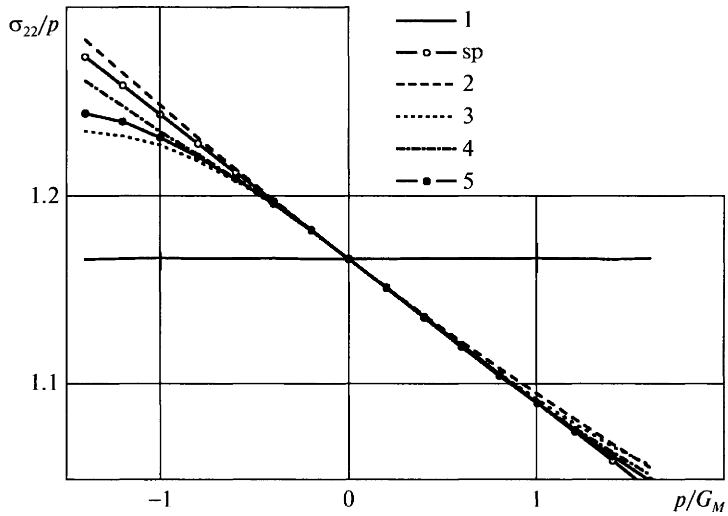

Fig. 4.

Problem 1. The simultaneous formation of two identical circular (at the instant of formation) inclusions of radius R. The matrix and inclusion materials are incompressible (of the Mooney type), $G_B/G_M = 5$, $\beta_B = \beta_M = 1$. It is assumed that the stresses in the inclusion depend on the overall strains (that is, the first version of the model for the formation of an inclusion is used). The initial load is a uniaxial extension along the x_2 axis: $(\sigma_{0,1})_{11} = (\sigma_{0,1})_2 = 0$, $(\sigma_{0,1})_{22} = p$. The centre of the first inclusion is located at the origin of coordinates.

The dependence of the overall true stresses $(\sigma_{0,2})_{22}$ in the matrix at point A with the coordinates $x_1 = 0$, $x_2 = R$ on the distance d between the centres of the inclusions (given at the instant of their formation) for the case when these centres are located on the x_2 axis is shown in Fig. 1, *a*: $p/G_M = 1$.

The dependence of the overall true stresses $(\sigma_{0,2})_{22}$ in the matrix at the same point on the distance d_1 between the centres of the inclusions in the direction of the x_1 axis, for the case when the distance between these centres in the direction of the x_2 axis is fixed ($d_2/R = 2.2$), that is, the centre of the second inclusion "moves" parallel to the x_1 axis; $p/G_M = 1.5$ is presented in Fig. 2.

Problem 2. The difference from Problem 1 lies solely in the fact that the stresses in the inclusion depend on the additional strains (the second version of the model for the formation of an inclusion is used).

The dependence analogous to that presented in Fig. 1, other conditions being equal, is shown for this case in Fig. 1, *b*. Comparing the results presented in Figs. 1, *a* and *b*, it can be seen that the stresses at one and the same point, when the different versions of the model for the formation of an inclusion are used, not only differ considerably in their absolute magnitude but also have different signs.

The dependence of the overall true stresses $(\sigma_{0,2})_{22}$ in the matrix at a point A with the coordinates $x_1 = R$, $x_2 = 0$ on the distances d between the centres of the inclusions (given at the instant of their formation) is shown in Fig. 3 for the case when these centres are located on the x_1 axis: $p/G_M = 1.5$.

Problem 3. The formation of a circular (in the final state) inclusion of radius R. The matrix and inclusion materials are incompressible (of the Mooney type), $G_B/G_M = 2$, $\beta_B = \beta_M = 1$. The initial load is a uniaxial extension along the x_2 axis: $(\sigma_{0,1})_{11} = (\sigma_{0,1})_{12} = 0$, $(\sigma_{0,1})_{22} = p$. The centre of the inclusion is located at the origin of coordinates. The calculation was performed using two methods: the small-parameter method (the first two approximations were found) and a modified Newton–Kantorovich (NK) method[31] (five approximations were calculated). The dependence of the stress concentration $(\sigma_{0,2})_{22}/p$ at the centre of the inclusion on the magnitude p of the initial load is shown in Fig. 4. The horizontal line corresponds to the solution obtained using of the linear theory of elasticity. The line marked with the small open circles corresponds to the solution obtained by the small-parameter method. The dashed line, the dotted line and the dot-dash line as well as the line marked by the small solid points correspond to the different approximations of the NK method. The numbers 2 - 5 indicate the number of the approximation.

It is of interest to investigate the following questions: 1) how large is the difference between the results obtained by the small-parameter method and the NK method?, 2) does the difference between two successive approximations of the NK method decrease as in the number of approximations increases?, 3) what is the correction when account is taken of non-linear effects in the small-parameter method? The results of a comparison of the solutions of this problem, obtained by the small-parameter method and the NK method, are shown in Fig. 5 for a different number of approximation's. The comparison was carried out for the stress components $(\sigma_{0,2})_{22}$. The following notation is used

$$\delta_{sp} = \max\left|\frac{(\sigma_{0,2})_{22}^{sp} - (\sigma_{0,2})_{22}^{1}}{(\sigma_{0,2})_{22}^{sp}}\right|, \quad \tilde{\delta} = \max\left|\frac{(\sigma_{0,2})_{22}^{sp} - (\sigma_{0,2})_{22}^{(5)}}{(\sigma_{0,2})_{22}^{(5)}}\right|$$

$$\delta_k = \max\left|\frac{(\sigma_{0,2})_{22}^{(k-1)} - (\sigma_{0,2})_{22}^{(k)}}{(\sigma_{0,2})_{22}^{(k)}}\right|, \quad k = 2,3,5$$

The superscript (k) indicates the approximation number for to the NK method, the superscript sp corresponds to the first approximation of the small-parameter method and the superscript 1 corresponds to the linear solution. In all cases, the maximum is determined in the interval $0 \le x_2 \le 2R$ of the x_2 axis (that is, when $x_1 = 0$). The graphs of the quantities δ_{sp}, $\tilde{\delta}$, δ_2, δ_3, δ_5 against the magnitude p of the initial load are given.

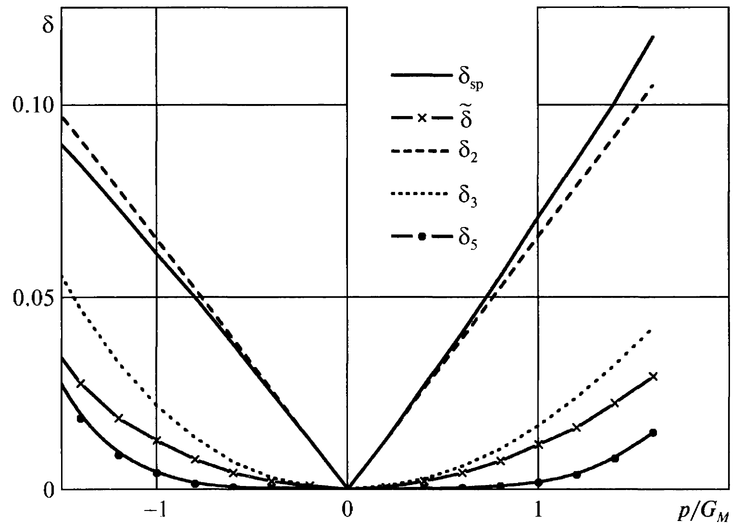

Fig. 5.

The stress distribution $(\sigma_{0,2})_{22}$ along the x_2 axis for the problem considered when $p/G_M = 1.5$ is shown in Fig. 6. The curve marked with the open circles corresponds to the solution obtained by the small-parameter method. The curves with the numbers 1 - 5 correspond to the different approximations of the NK method. The numbers 1 - 5 for these curves indicate the number of the approximation (note that the first approximation corresponds to the linear solution).

The data presented in Figs. 4 and 5 show that, when $-1.2p \leq /G_M \leq 1.4$, there is only a small difference between the fourth and fifth approximations of the NK method. The closeness of these approximations is also seen in Fig. 6. The difference between the results of the fourth and fifth approximations does not exceed 1%, which obviously enables us to conclude that the method converges over the above mentioned range of loads in the case of the version of the formulation of the problem considered for the values of the material constants given above. Note that, in the case of the results presented in Figs. 4 and 5, the difference between the solution obtained by the small-parameter method and the fifth approximation of the NK method does not exceed 2% and, in the case of the results presented in Fig. 6, this difference does not exceed 3%. At the same time, the correction when account is taken of non-linear effects in the small-parameter method reaches a value of 11%. We also note that, when $p/G_M = 1.5$, the relative elongation in the direction of the x_2 axis after the initial deformation $\lambda_2 \approx 1.41$. Hence, the deformations are not small in this case.

In the case of the problem of the formation of a hole, an approach to the analysis of the error in the small-parameter method and the NK method, is similar to what has been described, has been presented earlier.[31,41]

Problem 4. The formation of a circular domain with randomly distributed micropores in a body, the dimensions of which are small compared with its radius *R*. The properties of the material within this domain are described by effective constitutive relations constructed using the technique proposed earlier.[22,23,42,43] When performing the calculations, this domain is considered as a homogeneous inclusion, the

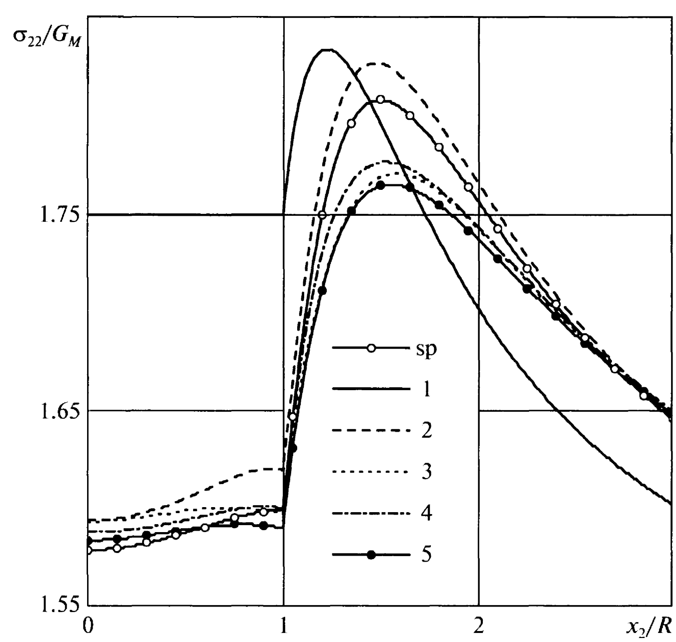

Fig. 6.

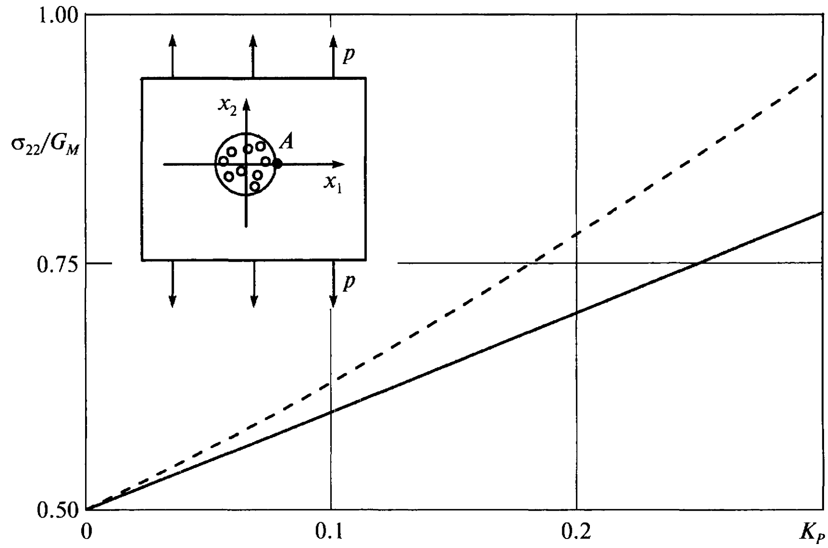

Fig. 7.

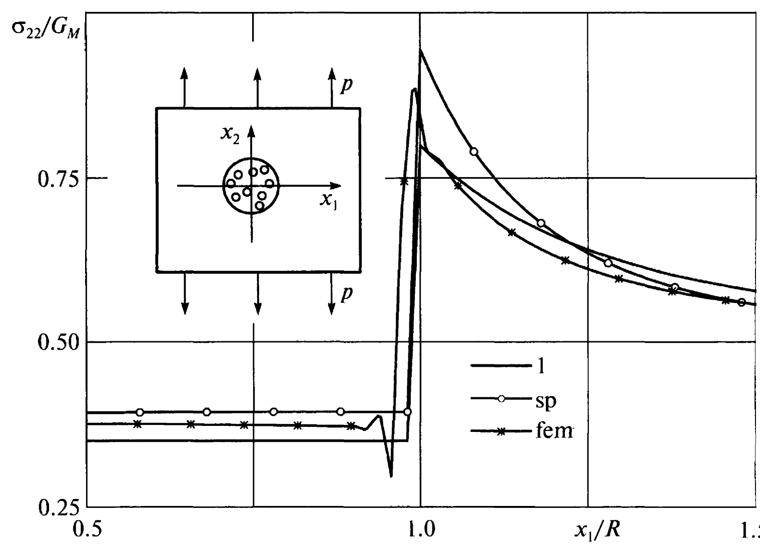

Fig. 8.

elastic moduli of which depend on the coefficient of porosity K_P. The first version of the model for the formation of an inclusion is used. The properties of the initial material are described by constitutive relations of the Murnaghan type with the following values of the constants:[44]

$$\lambda/G = 2.1, \quad C_3/G = -0.07, \quad C_4/G = -0.38, \quad C_5/G = 0.34$$

The initial load is a uniaxial extension in the direction of the x_2 axis: $p/G_M = 0.5$.

The dependence of the overall true stresses $(\sigma_{0,2})_{22}$ in the matrix at a point A with the coordinates $x_1 = R$, $x_2 = 0$ on the coefficient of porosity K_p is shown in Fig. 7. It is interesting to note that, in the case of the solution obtained using of the linear theory, this dependence is linear.

The stress distribution $(\sigma_{0,2})_{22}$ along the x_1 axis for the case when $K_P = 0.3$, obtained using linear theory (curve 1) and the small-parameter method (the curve sp) is shown in Fig. 8. Results of the solution of the same problem obtained by a finite element method (FEM),[25,45,46,47,48] based on the Bubnov method[49] (curve fem), are shown for comparison. A detailed investigation of the FEM, as applied to the problems considered here, is beyond the scope of this paper. We note that, when the FEM was used, the system of non-linear algebraic equations was solved by the unmodified NK method, that is, the matrix of the system of linear algebraic equations at each step in the NK method was constructed taking account of the solutions obtained in the preceding step. It is clear that there is a small difference between the finite element solution and the solution obtained by the small parameter method.

Acknowledgements

We wish to thank L. M. Zubov for useful conversations on the problems considered in this paper.

This research was financed by the Russian Foundation for Basic Research (06-01-00682-a).

References

1. Bartenev GM, Frenkel' SYa. *The Physics of Polymers*. Leningrad: Khimiya; 1990.
2. Bukhina MF. *Crystallization of Rubbers and Vulcanized Rubbers*. Moscow: Khimiya; 1973.
3. Bukhina MF, Zorina NM, Morozov YuL. Partially crystallized elastomer as a model of a nanocomposite. In: *Proc 15th Symposium "Problems of Tyres and Vulcanized Rubber - Cord Composites.* Moscow:GUP NII of Tyre Production;2005;Vol. 1; 74-82.
4. Kargin VA, Slonimskii GL. *Short Essays on the Physical Chemistry of Polymers*. Moscow: Khimiya; 1967.
5. Malkin AYa, Kulichikhin SG. Phase transitions in polymer system caused by the action of mechanical fields. *Vysokomolekulyarnye Soyedineniya Ser B* 1996;**38**(2):362–74.
6. Mandelkern L. *Crystallization of Polimers*. New York: McGraw-Hill; 1964.
7. Shur AM. *High Molecular Weight Compounds*. Moscow: Vyssh Shk; 1981.
8. Boyce MC, Scorate S, Llana PG. Constitutive model for t he finite deformation stress–strain behavior of poly(ethylene terephthalate) above the glass transition. *Polymer* 2000;**41**(6):2183–201.
9. Drozdov AD. Mechanically induced crystallization of polymers. *Intern J Non-Linear Mech* 1999;**34**(5):807–21.
10. Negahban M. Modeling the thermomechanical effects of crystallization in natural rubber: I. The theoretical structure. *Intern J Solids and Structures* 2000;**37**(20):2777–89.
11. Oshmyan VG, Timan SA, Shamayev MYu. Modelling of the effect of the structure of an amorphous-crystalline polymer on its deformation properties. *Vysokomolekulyarnye Soyedineniya Ser A* 2003;**45**(10):1699–706.
12. Oshmyan VG, Patlazhan SA, Remond Y. Effect of structural changes and non-linearity of plastic flow on small deformations of partially crystallized polymers. *Vysoko-molekulyarnye Soyedineniya Ser A* 2005;**47**(4):600–7.
13. Grinfeld MA. *Methods of continuum Mechanics in the Theory of Phase Transitions*. Moscow: Nauka; 1990.
14. Yeremeyer VA, Sotnichenko DM. Some problems of phase transactions in deformed media at finite strains. *Izv Vuzov Severo-Kavkaz Region Yestestv Nauki Spetsvypusk "Nelineinye Problemy Mekhaniki Sploshnykh Sred* 2000:52–74.
15. Konduarov VI. Kinetics of phase transitions of the first kind in a thermoelastic material. *Dokl Rass Akad Nauk* 2004;**396**(2):194–8.
16. Levimtas VI. *Thermodynamics of Phase Transitions and the Inelastic Deformation of Micro- Inhomogeneous Materials*. Kiev: Naukova Dumka; 1992.
17. Levitas VI, Preston DL. Thermomechanical lattice instability and phase field theory of martensitic phase transformations, twinning and dislocations at large strains. *Phys Letters* 2005;**A343**(1–3):32–9.
18. Movchan AA, Sil'chenko LG. The stability of a plate of shape-memory alloy in a direct thermoelastic phase transition. *Prikl Mat Mekh* 2004;**68**(1):60–72.
19. Levin VA, Morozov YeM. Non-local criteria for the determining of a preexisting imperfection zone in describing the growth of a defect at finite strains. *Dokl Rass Akad Nauk* 2007;**415**(1):52–4.
20. Vavakin AS, Salganik RL. On the effective characteristics of inhomogeneous media with isolated inhomogeneities. *Izv Akad Nauk SSSR MTT* 1975;**3**:65–75.
21. Kachanov M, Tsukrov L, Shafiro B. Effective moduli of a solid with cavities of various shapes. *Appl Mech Reviews* 1994;**47**(1 Pt 2):S151–74.
22. Levin VA. *Multiple Superpositioning of Large Deformations in Elastic and Viscoelastic Bodies*. Moscow: Nauka; 1999.
23. Levin VA, Zingerman KM. On the construction of effective constitutive relations for porous elastic materials at finite strains and their superpositioning. *Dokl Rass Akad Nauk* 2002;**382**(4):482–7.
24. Levin VA, Morozov YeM, Matviyenko YuG. In: Levin VA, editor. *Selected Non-Linear Problems of Fracture Mechanics*. Moscow: Fizmatlit; 2004.
25. Levin VA, Kalinin VV, Zingerman KM, Vershinin AV. *The Development of Defects at Finite Strains. Computer and Physical Modelling*. Moscow: Fizmatlit; 2007.
26. Eshelby J. *Determination of the elastic stress fields created by an ellipsoidal inclusion and problems associated with this problem*. In: *Continuum Theory of Dislocations*. Moscow: Izd Inostr Lit; 1963, 103–109.
27. Muskhelishvili NI. *Some Basic Problems in the Mathematical Theory of Elasticity*. Amsterdam: Klumer; 1977.
28. Tsvelodub IYu. A physically non-linear inclusion in a linear elastic medium (the plane problem). *Izv Akad Nauk MTT* 2000;**5**:72–84.
29. Chernykh KF. Some comments on the Eshelby problem. *Izv Ross Akad Nauk MTT* 1994;**4**:47–50.
30. Morozov NF, Nazyrov IR, Freidin AB. The one-dimensional problem of the phase transition of an elastic sphere. *Dokl Ross Akad Nauk* 1996;**346**(2):188–91.
31. Levin VA, Zingerman KM. *Plane Problems in the Theory of Multiple Superpositioning of Large Deformations. Methods of Solution*. Moscow: Nauka; 2002.
32. Levin VA, Taras'ev GS. Superposition of large elastic deformations in the space of final states. *Dokl Akad Nauk SSSR* 1980;**251**(1):63–6.
33. Lurie AI. *Non linear Theory of Elasticity*. Amsterdam: North-Holland; 1990.
34. Freidin AB. The phase equilibrium of a non-linear elastic material. *Izv Vuzov Severo- Kavkaz Region estestv Nauki Spetsvypusk "Nelineinye Problemy Mekhaniki Sploshnykh Sred* 2000:150–68.
35. Savin GN. *Stress Concentration Around Holes*. Oxford: Pergamon; 1961.
36. Kantorovich LV, Akilov GP. *Functional Analysis*. Oxford: Pergamon; 1982.
37. Taras'ev GS. The estimation of the "small" parameter in a problem of the non-linear theory of elasticity. *Prikl Mekh* 1980;**16**(7):137–9.
38. Svistkov AL, Yevlamniyva SYe. An Iteration method for the calculating the stress-strain state in ensembles of inclusions. In: *Structural Mechanisms of the Formation of the Mechanical Properties of Granular Polymer Composites*. 1997. p. 171–203.
39. Sedov LI. *Continuum Mechanics*, Vol.2. Moscow: Nauka; 1994.
40. Zingerman KM. The solution of a plane problem in the theory of elasticity for a body with several circular elastic inclusions at finite strains. *Izv Tul'skogo Gos Univ Seriya Matematika Mekhanika Informatika Vol. 12. Issue 2 Mekhanika.* Tula:Izd Tul's Kogo Gos Univ;2006;40-53.
41. Levin VA, Zingerman KM. The error in the approximate solution of the problem of the formation of a stress concentrator in a preloaded elastic body. Superposition of large deformations. *Izv Vuzov Severo-Kavkaz Region Yestestv Nauki Spetsvypusk "Nelineinye Problemy Mekhnaniki Sploshnoi Sredy".* 2000:99-106.
42. Levin VA, Lokhin VV, Zingerman KM. Effective elastic properties of porous materials with randomly dispersed pores. Finite deformation. *Trans ASME J Appl Mech* 2000;**67**(4):667–70.
43. Levin VA, Zingermann KM. Effective constitutive equations for porous elastic materials at finite strains and superimposed finite strains. *Trans ASME J Appl Mech* 2003;**70**(6):809–16.
44. Guz' AN, Makhort FG, Gushcha OI. *Introduction to Acoustic Elasticity*. Kiev: Naukova Dumka; 1977.
45. Marchuk GI. *Methods of Computational Mathematics*. Moscow: Nauka; 1989.
46. Pobedrya BYe. *Numerical Methods in the Theory of Elasticity and Plasticity*. Moscow: Izd MGU; 1995.
47. Zienkiewicz OC, Taylor RL. *The Finite Element Method. V. 2. Solid mechanics*. Butterworth-Heinemann; 2000, 384 p.
48. Levin VA, Zingerman KM, Vershinin AV. Comparison of the analytical and numerical solutions of the problem of the stress distribution close to inclusions and cavities formed in a loaded non-linear elastic body with the superposition of finite strains. In: *Grid Methods for Boundary-Value Problems and Applications. Proceeding of the 6th All- Russian Seminar.* Kazan: Kazan Gos Univ;2005;157-60.
49. Bakhvalov NS, Zhidkov NP, Kobel'kov GM. *Numerical Methods*. Moscow: Nauka; 1987.

Translated by E.L.S.